Lesson 6.5A Working with Radicals

Activity 1 Equivalent Radicals

We use the **product and quotient rules for radicals** to **simplify radicals**. To "simplify" a radical does **not** mean to find a decimal approximation using a calculator!! The new expression is an **exact equivalent** for the original radical, not an approximation.

1. Use your calculator to verify each calculation.

a. $\sqrt{12} = \sqrt{\mathbf{4} \cdot 3} = \sqrt{\mathbf{4}}\sqrt{3} = \mathbf{2}\sqrt{3}$ (Note that $2\sqrt{3}$ means 2 times $\sqrt{3}$.)

b. $\sqrt{45} = \sqrt{\mathbf{9} \cdot 5} = \sqrt{\mathbf{9}}\sqrt{5} = \mathbf{3}\sqrt{5}$

c. $\sqrt[3]{40} = \sqrt[3]{\mathbf{8} \cdot 5} = \sqrt[3]{\mathbf{8}} \cdot \sqrt[3]{5} = \mathbf{2}\sqrt[3]{5}$

d. $\sqrt[4]{162} = \sqrt[4]{\mathbf{81} \cdot 2} = \sqrt[4]{\mathbf{81}} \cdot \sqrt[4]{2} = \mathbf{3}\sqrt[4]{2}$

2. There is a difference between simplifying a radical and finding a decimal approximation for a radical! Compare:

Simplify $\sqrt{45}$ Answer: $\sqrt{45} = \sqrt{\mathbf{9}}\sqrt{5} = \mathbf{3}\sqrt{5}$ (no calculators!)

Approximate $\sqrt{45}$ Answer: $\sqrt{45} \approx 6.708$ (by calculator)

3. Now you simplify some. Look for a **perfect square** that divides evenly into the radicand:

a. $\sqrt{200} =$

b. $\sqrt{48} =$

Look for a **perfect cube** that divides evenly into the radicand:

c. $\sqrt[3]{72} =$

d. $\sqrt[3]{250} =$

Activity 2 Variables under the Radical

1. We can do this with variables, too. For example:

$\sqrt{x^4} = x^2$ because $(x^2)^2 = x^4$

$\sqrt{x^{10}} = x^5$ because $(x^5)^2 = x^{10}$

a. $\sqrt{x^6} =$ _______ because $(\_\_\_\_)^2 = x^6$

b. $\sqrt[3]{x^6} =$ _______ because $(x^2)^3 = x^6$

c. $\sqrt[3]{x^{15}} =$ _______ because $(\_\_\_\_)^3 = x^{15}$

d. $\sqrt[4]{x^{12}} =$ _______ because $(\_\_\_\_)^4 = x^{12}$

2. Now we'll simplify some roots of products. Finish each example:

a. $\sqrt{81\,x^8} = \sqrt{81}\sqrt{x^8} =$

b. $\sqrt{16\,x^{16}} = \sqrt{16}\sqrt{x^{16}} =$

c. $\sqrt[3]{8x^9} = \sqrt[3]{8}\sqrt[3]{x^9} =$

d. $\sqrt[4]{81\,x^{20}} =$

3. Here are some roots of quotients. Finish each example:

a. $\sqrt{\dfrac{9a^8}{16b^4}} = \dfrac{\sqrt{9a^8}}{\sqrt{16b^4}} =$

b. $\sqrt[3]{\dfrac{-27y^9}{125z^6}} =$

Activity 3 Simplifying Radicals

1. What about simplifying the square root of an odd power? An even power is a perfect square, so we factor off one power of x. For example,

$$x^7 = \boldsymbol{x^6} \cdot x \qquad \text{and} \qquad x^{13} = \boldsymbol{x^{12}} \cdot x$$

a. $\sqrt{x^7} = \sqrt{x^6}\sqrt{x} = x^3\sqrt{x}$ (Note that $x^3\sqrt{x}$ means x^3 times $\sqrt{x}$.)

b. $\sqrt{x^{13}} = \sqrt{x^{12}}\sqrt{x} =$

For cube roots, we factor out a power whose exponent is divisible by three. For example,

$$x^8 = \boldsymbol{x^6} \cdot x^2 \qquad \text{and} \qquad x^{13} = \boldsymbol{x^{12}} \cdot x$$

c. $\sqrt[3]{x^8} = \sqrt[3]{x^6} \cdot \sqrt[3]{x^2} = x^2\ \sqrt[3]{x^2}$ (We cannot simplify $\sqrt[3]{x^2}$ any further.)

d. $\sqrt[3]{x^{13}} = \sqrt[3]{x^{12}} \cdot \sqrt[3]{x} =$

2. Now we can simplify any radical. To simplify a square root, factor out any perfect squares from the radicand. Factor the numbers and each variable separately.

a. $\sqrt{12t^5} = \sqrt{4t^4 \cdot 3t} = \sqrt{4t^4}\sqrt{3t} =$ ______ $\cdot \sqrt{3t}$

b. $\sqrt{27w^9} = \sqrt{9w^8 \cdot 3w} = \sqrt{9w^8}\sqrt{3w} =$ _____________

c. $\sqrt{72b^{16}c^{15}} =$

To simplify a cube root, factor out any perfect cubes from the radicand.

d. $\sqrt[3]{32m^{11}} = \sqrt[3]{8m^9}\ \sqrt[3]{4m^2} =$ ______ $\cdot \sqrt[3]{4m^2}$

e. $\sqrt[3]{81a^{12}b^8} = \sqrt[3]{27a^{12}b^6}\ \sqrt[3]{3b^2} =$ _______________

f. $\sqrt[3]{40x^{14}y^9} =$

3. Now you try some. Simplify:

a. $\sqrt{18xy^3} =$

b. $\sqrt{80\, a^5b^4} =$

c. $\sqrt[3]{24\, m^3p^2} =$

d. $\sqrt[3]{64\, w^4z^2} =$

Lesson 6.5B Working with Radicals

Activity 1 Products and Quotients of Radicals

1. We can combine the radicands in products or quotients of radicals, as long as both radicals have the same **index**. For example:

a. $\sqrt{12x}\sqrt{3x} = \sqrt{(12x)(3x)} = \sqrt{36x^2} = 6x$

b. $\sqrt[4]{2m^3}\sqrt[4]{8m} = \sqrt[4]{(2m^3)(8m)} = \sqrt[4]{\quad ? \quad} =$ ____________

2. Complete the following examples of quotients:

a. $\dfrac{\sqrt{18xy^3}}{\sqrt{2y}} = \sqrt{\dfrac{18xy^3}{2y}} =$

b. $\dfrac{\sqrt[3]{200a^8}}{\sqrt[3]{5a^2}} = \sqrt[3]{\dfrac{200a^8}{5a^2}} =$

Activity 2 Sums and Differences of Radicals

1. Recall that we can combine radicals by multiplying and dividing, but not by adding and subtracting. For example,

$$\sqrt{12}\sqrt{3} = \sqrt{36} \qquad \text{and} \qquad \frac{\sqrt{12}}{\sqrt{3}} = \sqrt{4}$$

$$\textbf{but} \quad \sqrt{12} + \sqrt{3} \neq \sqrt{15} \qquad \text{and} \qquad \sqrt{12} - \sqrt{3} \neq \sqrt{9}$$

(You should check these on your calculator.) Decide whether each of the following is true or false (use your calculator):

a. $\sqrt{7} + \sqrt{7} = \sqrt{14}$?

b. $\sqrt{7} + \sqrt{7} = 2\sqrt{7}$?

c. $\sqrt{7} - \sqrt{5} = \sqrt{2}$?

d. $6\sqrt{5} - 2\sqrt{5} = 4\sqrt{5}$?

2. We can only add or subtract **like radicals**, that is, radicals with the same index and the same radicand. This is similar to adding or subtracting like terms. Compare:

$3\sqrt{7} + 2\sqrt{7} = 5\sqrt{7}$	and	$3x + 2x = 5x$
$8\sqrt{5} - 3\sqrt{2}$ c.b.s.	and	$8x - 3y$ c.b.s.
$4\sqrt{x} + 3\sqrt{xy}$ c.b.s.	and	$4x + 3xy$ c.b.s.
$\sqrt{5} - \sqrt{5x}$ c.b.s.	and	$5 - 5x$ c.b.s.

Notice that the radicand does not change when we add or subtract like radicals, only the coefficients are combined. (The same is true when we add or subtract like terms.)

True or False:

a. $a\sqrt{5}+a\sqrt{2}=a\sqrt{7}$?

b. $\sqrt{11}+\sqrt[3]{11}=\sqrt[5]{11}$?

c. $2x\sqrt{3x}+5\sqrt{3x}=7x\sqrt{6x}$?

d. $\sqrt[3]{25}-\sqrt[3]{15}=\sqrt[3]{10}$?

3. Now you try some. Simplify by combining like terms:

a. $5\sqrt{6}-2\sqrt{6}+6\sqrt{2}=$

b. $8\sqrt[3]{a^2}-5\sqrt[3]{a}-5\sqrt[3]{a^2}=$

4. Sometimes we have to simplify the radicals before we see that they are like radicals:

$$\begin{aligned}\sqrt{8y}-\sqrt{18y}&=\sqrt{\mathbf{4}\cdot 2y}-\sqrt{\mathbf{9}\cdot 2y} && \text{Simplify each radical.}\\ &=\sqrt{\mathbf{4}}\sqrt{2y}-\sqrt{\mathbf{9}}\sqrt{2y}\\ &=\mathbf{2}\sqrt{2y}-\mathbf{3}\sqrt{2y}=-\sqrt{2y} && \text{Combine like radicals.}\end{aligned}$$

Simplify and combine like terms:

a. $2\sqrt{75}+3\sqrt{48}=$

b. $\sqrt[3]{81}+2\sqrt[3]{24}-3\sqrt[3]{3}=$

Activity 3 Using the Distributive Law

1. First, consider some products that do not involve the distributive law. Simplify each product as much as possible.

a. $2\sqrt{5}=$

b. $\sqrt{2}\sqrt{5}=$

c. $\sqrt{5}\sqrt{5}=$

d. $(3\sqrt{2})(2\sqrt{5})=$

e. $(3\sqrt{5})(2\sqrt{5})=$

f. $(2\sqrt{15})(3\sqrt{6})=$

2. Apply the distributive law to compute the products:

a. $5(2\sqrt{3} - \sqrt{5})$

b. $4\sqrt{2}\,(4\sqrt{3} + 2\sqrt{6})$

c. $(3\sqrt{x} - \sqrt{5})\,(2\sqrt{x} + 3\sqrt{5})$

Activity 4 Rationalizing the Denominator

1. In some situations, it is not convenient to leave a radical in the denominator of a fraction. Explain why each of the following pairs are equal:

a. $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ **b.** $\dfrac{5}{\sqrt{2}} = \dfrac{5\sqrt{2}}{2}$

(Hint: Multiply top and bottom of the first fraction by the same number to get the second fraction.) Use your calculator to verify that the fractions are equal.

2. Converting a fraction to an equivalent form with no radicals in the denominator is called **rationalizing the denominator**. (It's okay to have radicals in the numerator.) For each fraction, multiply top and bottom by a number that will rationalize the denominator.

a. $\dfrac{-\sqrt{3}}{\sqrt{7}} =$ **b.** $\dfrac{10}{\sqrt{5}} =$

3. Be careful: it is easier to simplify the denominator before you rationalize it! For example:

$$\frac{6}{\sqrt{12}} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}}$$

Now rationalize the denominator:

$$\frac{3}{\sqrt{3}} \cdot \frac{\boldsymbol{\sqrt{3}}}{\boldsymbol{\sqrt{3}}} =$$

(Or better yet, recognize that $\dfrac{3}{\sqrt{3}} = \sqrt{3}$ by the definition of square root.)

a. $\sqrt{\dfrac{27x}{20}} =$

b. $\dfrac{6\sqrt{2}}{\sqrt{3v}} =$

4. To rationalize a binomial denominator, we use the **conjugate** of the binomial. For example:

Binomial	**Conjugate**
$3+\sqrt{7}$	$3-\sqrt{7}$
$\sqrt{6}-\sqrt{3}$	$\sqrt{6}+\sqrt{3}$
$\sqrt{5}+2$	?

Try multiplying each binomial above by its conjugate and see what happens:

a. $(3+\sqrt{7})\,(3-\sqrt{7}\,) =$

b. $(\sqrt{6}-\sqrt{3})(\sqrt{6}+\sqrt{3}) \;=$

c. $(\sqrt{5}+2)\,($? $) \;=$

No more radicals, right? Also, did you notice that each calculation is of the form $(a+b)(a-b)$? So you know that the product will be a^2-b^2.

5. a. Now we'll use the conjugate to rationalize a binomial denominator. Consider the fraction $\dfrac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}$. We multiply top and bottom of the fraction by the conjugate of the denominator:

$$\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}\cdot\frac{\boldsymbol{\sqrt{3}+\sqrt{2}}}{\boldsymbol{\sqrt{3}+\sqrt{2}}} = \frac{\sqrt{3}\,\sqrt{3}+\sqrt{3}\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}$$
$$= \frac{3+\sqrt{6}}{3-2} = 3+\sqrt{6}$$

Now try some yourself:

b. $\dfrac{y}{\sqrt{5}-y}$

c. $\dfrac{\sqrt{6}-3}{2-\sqrt{6}}$

d. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

Wrap-Up

In this Lesson, we worked on the following skills and goals related to powers and roots:

- Simplifying radicals
- Adding or subtracting like radicals
- Simplifying products or quotients of radicals
- Rationalizing the denominator

Check Your Understanding

1. Explain the difference between simplifying a radical and evaluating a radical.

2. Explain how to simplify a cube root. List the cubes of the first six whole numbers.

3. Compare the rules for adding radicals and multiplying radicals. Illustrate by computing $4\sqrt[3]{2x^2}+3\sqrt[3]{2x^2}$ and $4\sqrt[3]{2x^2}\cdot 3\sqrt{2x^2}$.

4. Explain why we use a conjugate to rationalize a binomial denominator.