Addition and Subtraction of Rational Expressions

5.3

This section is concerned with addition and subtraction of rational expressions. In the first part of this section, we will look at addition of expressions that have the same denominator. In the second part of this section, we will look at addition of expressions that have different denominators.

Addition and Subtraction with the Same Denominator

To add two expressions that have the same denominator, we simply add numerators and put the sum over the common denominator. Because the process we use to add and subtract rational expressions is the same process used to add and subtract fractions, we will begin with an example involving fractions.

EXAMPLE 1 Add $\frac{4}{9} + \frac{2}{9}$.

SOLUTION We add fractions with the same denominator by using the distributive property. Here is a detailed look at the steps involved.

$$\begin{aligned}
\frac{4}{9} + \frac{2}{9} &= 4\left(\frac{1}{9}\right) + 2\left(\frac{1}{9}\right) \\
&= (4 + 2)\left(\frac{1}{9}\right) && \text{Distributive property} \\
&= 6\left(\frac{1}{9}\right) \\
&= \frac{6}{9} \\
&= \frac{2}{3} && \text{Divide numerator and denominator by common factor 3.}
\end{aligned}$$

Note that the important thing about the fractions in this example is that they each have a denominator of 9. If they did not have the same denominator, we could not have written them as two terms with a factor of $\frac{1}{9}$ in common. Without the $\frac{1}{9}$ common to each term, we couldn't apply the distributive property. Without the distributive property, we would not have been able to add the two fractions.

In the following examples, we will not show all the steps we showed in Example 1. The steps are shown in Example 1 so you will see why both fractions must have the same denominator before we can add them. In practice, we simply add numerators and place the result over the common denominator.

We add and subtract rational expressions with the same denominator by combining numerators and writing the result over the common denominator. Then we reduce the result to lowest terms, if possible. Example 2 shows this process in detail. If you see the similarities between operations on rational numbers and operations on rational expressions, this chapter will look like an extension of rational numbers rather than a completely new set of topics.

EXAMPLE 2 Add $\dfrac{x}{x^2-1} + \dfrac{1}{x^2-1}$.

SOLUTION Because the denominators are the same, we simply add numerators:

$$\frac{x}{x^2-1} + \frac{1}{x^2-1} = \frac{x+1}{x^2-1}$$ Add numerators.

$$= \frac{\cancel{x+1}}{(x-1)\cancel{(x+1)}}$$ Factor denominator.

$$= \frac{1}{x-1}$$ Divide out common factor x + 1

Our next example involves subtraction of rational expressions. Pay careful attention to what happens to the signs of the terms in the numerator of the second expression when we subtract it from the first expression.

EXAMPLE 3 Subtract $\dfrac{2x-5}{x-2} - \dfrac{x-3}{x-2}$.

SOLUTION Because each expression has the same denominator, we simply subtract the numerator in the second expression from the numerator in the first expression and write the difference over the common denominator $x - 2$. We must be careful, however, that we subtract both terms in the second numerator. To ensure that we do, we will enclose that numerator in parentheses.

$$\frac{2x-5}{x-2} - \frac{x-3}{x-2} = \frac{2x-5-(x-3)}{x-2}$$ Subtract numerators.

$$= \frac{2x-5-x+3}{x-2}$$ Remove parentheses.

$$= \frac{\cancel{x-2}}{\cancel{x-2}}$$ Combine similar terms in the numerator.

$$= 1$$ Reduce (or divide).

Note the $+3$ in the numerator of the second step. It is a common mistake to write this as -3, by forgetting to subtract both terms in the numerator of the second expression. Whenever the expression we are subtracting has two or more terms in its numerator, we have to watch for this mistake.

Next we consider addition and subtraction of fractions and rational expressions that have different denominators.

Addition and Subtraction With Different Denominators

Before we look at an example of addition of fractions with different denominators, we need to review the definition for the least common denominator (LCD).

DEFINITION *least common denominator*

The ***least common denominator*** for a set of denominators is the smallest expression that is divisible by each of the denominators.

The first step in combining two fractions is to find the LCD. Once we have the common denominator, we rewrite each fraction as an equivalent fraction with the common denominator. After that, we simply add or subtract as we did in our first three examples.

Example 4 is a review of the step-by-step procedure used to add two fractions with different denominators.

EXAMPLE 4 Add $\frac{3}{14} + \frac{7}{30}$.

SOLUTION

Step 1: *Find the LCD.*

To do this, we first factor both denominators into prime factors.

Factor 14: $14 = 2 \cdot 7$

Factor 30: $30 = 2 \cdot 3 \cdot 5$

Because the LCD must be divisible by 14, it must have factors of $2 \div 7$. It must also be divisible by 30 and, therefore, have factors of $2 \div 3 \div 5$. We do not need to repeat the 2 that appears in both the factors of 14 and those of 30. Therefore,

$$\text{LCD} = 2 \cdot 3 \cdot 5 \cdot 7 = 210$$

Step 2: *Change to equivalent fractions.*

Because we want each fraction to have a denominator of 210 and at the same time keep its original value, we multiply each by 1 in the appropriate form.

Change 3/14 to a fraction with denominator 210:

$$\frac{3}{14} \cdot \frac{15}{15} = \frac{45}{210}$$

Change 7/30 to a fraction with denominator 210:

$$\frac{7}{30} \cdot \frac{7}{7} = \frac{49}{210}$$

Step 3: *Add numerators of equivalent fractions found in step 2:*

$$\frac{45}{210} + \frac{49}{210} = \frac{94}{210}$$

Step 4: *Reduce to lowest terms, if necessary:*

$$\frac{94}{210} = \frac{47}{105}$$

The main idea in adding fractions is to write each fraction again with the LCD for a denominator. In doing so, we must be sure not to change the value of either of the original fractions.

EXAMPLE 5 Add $\dfrac{-2}{x^2 - 2x - 3} + \dfrac{3}{x^2 - 9}$.

SOLUTION

Step 1: *Factor each denominator and build the LCD from the factors:*

$$x^2 - 2x - 3 = (x - 3)(x + 1) \qquad x^2 - 9 = (x - 3)(x + 3) \qquad LCD = (x - 3)(x + 3)(x + 1)$$

Step 2: *Change each rational expression to an equivalent expression that has the LCD for a denominator:*

$$\frac{-2}{x^2 - 2x - 3} = \frac{-2}{(x - 3)(x + 1)} \cdot \frac{(x + 3)}{(x + 3)} = \frac{-2x - 6}{(x - 3)(x + 3)(x + 1)}$$

$$\frac{3}{x^2 - 9} = \frac{3}{(x - 3)(x + 3)} \cdot \frac{(x + 1)}{(x + 1)} = \frac{3x + 3}{(x - 3)(x + 3)(x + 1)}$$

Step 3: *Add numerators of the rational expressions found in step 2:*

$$\frac{-2x - 6}{(x - 3)(x + 3)(x + 1)} + \frac{3x + 3}{(x - 3)(x + 3)(x + 1)} = \frac{x - 3}{(x - 3)(x + 3)(x + 1)}$$

Step 4: *Reduce to lowest terms by dividing out the common factor $x - 3$:*

$$\frac{\cancel{x - 3}}{\cancel{(x - 3)}(x + 3)(x + 1)} = \frac{1}{(x + 3)(x + 1)}$$

EXAMPLE 6 Subtract $\dfrac{x + 4}{2x + 10} - \dfrac{5}{x^2 - 25}$.

SOLUTION We begin by factoring each denominator:

$$\frac{x + 4}{2x + 10} - \frac{5}{x^2 - 25} = \frac{x + 4}{2(x + 5)} - \frac{5}{(x + 5)(x - 5)}$$

The LCD is $2(x + 5)(x - 5)$. Completing the problem, we have

$$= \frac{x + 4}{2(x + 5)} \cdot \frac{(x - 5)}{(x - 5)} - \frac{5}{(x + 5)(x - 5)} \cdot \frac{2}{2}$$

$$= \frac{x^2 - x - 20}{2(x + 5)(x - 5)} - \frac{10}{2(x + 5)(x - 5)}$$

$$= \frac{x^2 - x - 30}{2(x + 5)(x - 5)}$$

To see if this expression will reduce, we factor the numerator into $(x - 6)(x + 5)$.

$$= \frac{(x - 6)\cancel{(x + 5)}}{2\cancel{(x + 5)}(x - 5)}$$

$$= \frac{x - 6}{2(x - 5)}$$

EXAMPLE 7 Subtract $\frac{2x-2}{x^2+4x+3} - \frac{x-1}{x^2+5x+6}$.

SOLUTION We factor each denominator and build the LCD from those factors:

$$\begin{aligned}
&\frac{2x-2}{x^2+4x+3} - \frac{x-1}{x^2+5x+6} \\
&= \frac{2x-2}{(x+3)(x+1)} - \frac{x-1}{(x+3)(x+2)} \\
&= \frac{2x-2}{(x+3)(x+1)} \cdot \frac{(x+2)}{(x+2)} - \frac{x-1}{(x+3)(x+2)} \cdot \frac{(x+1)}{(x+1)} && \text{The LCD is } (x+1)(x+2)(x+3) \\
&= \frac{2x^2+2x-4}{(x+1)(x+2)(x+3)} - \frac{x^2-1}{(x+1)(x+2)(x+3)} && \text{Multiply out each numerator.} \\
&= \frac{(2x^2+2x-4)-(x^2-1)}{(x+1)(x+2)(x+3)} && \text{Subtract numerators.} \\
&= \frac{x^2+2x-3}{(x+1)(x+2)(x+3)} && \text{Factor numerator to see if we can rdeuce.} \\
&= \frac{(x+3)(x-1)}{(x+1)(x+2)(x+3)} && \text{Reduce.} \\
&= \frac{x-1}{(x+1)(x+2)}
\end{aligned}$$

EXAMPLE 8 Add $\frac{x^2}{x-7} + \frac{6x+7}{7-x}$.

SOLUTION In Section 5.1, we were able to reverse the terms in a factor such as $7-x$ by factoring -1 from each term. In a problem like this, the same result can be obtained by multiplying the numerator and denominator by -1:

$$\begin{aligned}
\frac{x2}{x-7} + \frac{6x+7}{7-x} \cdot \frac{-1}{-1} &= \frac{x2}{x-7} + \frac{-6x-7}{x-7} \\
&= \frac{x^2-6x-7}{x-7} && \text{Add numerators} \\
&= \frac{\cancel{(x-7)}(x+1)}{\cancel{(x-7)}} && \text{Factor numerator.} \\
&= x+1 && \text{Divide out } x-7
\end{aligned}$$

For our next example, we will look at a problem in which we combine a whole number and a rational expression.

EXAMPLE 9 Subtract $2 - \frac{9}{3x + 1}$.

SOLUTION To subtract these two expressions, we think of 2 as a rational expression with a denominator of 1.

$$2 - \frac{9}{3x + 1} = \frac{2}{1} - \frac{9}{3x + 1}$$

The LCD is $3x + 1$. Multiplying the numerator and denominator of the first expression by $3x + 1$ gives us a rational expression equivalent to 2, but with a denominator of $3x + 1$.

$$\begin{aligned} \frac{2}{1} \cdot \frac{(3x + 1)}{(3x + 1)} - \frac{9}{3x + 1} &= \frac{6x + 2 - 9}{3x + 1} \\ &= \frac{6x - 7}{3x + 1} \end{aligned}$$

The numerator and denominator of this last expression do not have any factors in common other than 1, so the expression is in lowest terms.

EXAMPLE 10 Write an expression for the sum of a number and twice its reciprocal. Then, simplify that expression.

SOLUTION If x is the number, then its reciprocal is $\frac{1}{x}$. Twice its reciprocal is $\frac{2}{x}$. The sum of the number and twice its reciprocal is

$$x + \frac{2}{x}$$

To combine these two expressions, we think of the first term x as a rational expression with a denominator of 1. The LCD is x:

$$\begin{aligned} x + \frac{2}{x} &= \frac{x}{1} + \frac{2}{x} \\ &= \frac{x}{1} \cdot \frac{x}{x} + \frac{2}{x} \\ &= x^2 + \frac{2}{x} \end{aligned}$$

GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

A. Briefly describe how you would add two rational expressions that have the same denominator.

B. Why is factoring important in finding a least common denominator?

C. What is the last step in adding or subtracting two rational expressions?

D. Explain how you would change the fraction $\frac{5}{x - 3}$ to an equivalent fraction with denominator $x^2 - 9$.