5.7 The Pythagorean Theorem

Objectives

A Using the Pythagorean Theorem to solve right triangles.

B Solving 30°–60°–90° Triangles.

C Solving 45°–45°–90° Triangles.

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A Pythagorean Theorem

In any right triangle, the square of the length of the longest side (called the hypotenuse) is equal to the sum of the squares of the lengths of the other two sides (called legs).

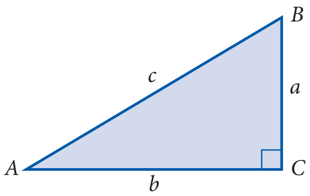


FIGURE 1

If $C = 90°$,
then $c^2 = a^2 + b^2$

Next we will prove the Pythagorean Theorem. Part of the proof involves finding the area of a triangle. In any triangle, the area is given by the formula

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$$

For the right triangle shown in Figure 1, the base is b, and the height is a. Therefore the area is $A = \frac{1}{2}ab$.

There are many ways to prove the Pythagorean Theorem. The method that we are offering here is based on the diagram shown in Figure 2 and the formula for the area of a triangle.

Figure 2 is constructed by taking the right triangle in the lower right corner and repeating it three times so that the final diagram is a square in which each side has length $a + b$.

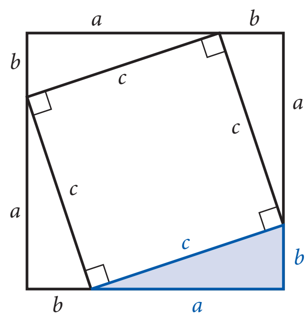


FIGURE 2

To derive the relationship between a, b, and c, we simply notice that the area of the large square is equal to the sum of the areas of the four triangles and the inner square. In symbols we have

Area of large square		Area of four triangles		Area of inner square
$(a + b)^2$	$=$	$4\left(\frac{1}{2}ab\right)$	$+$	c^2

We expand the left side using the formula from algebra for the square of a binomial. We simplify the right side by multiplying 4 with $\frac{1}{2}$.

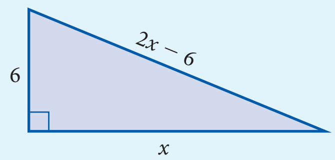
$$a^2 + 2ab + b^2 = 2ab + c^2$$

Adding $-2ab$ to each side, we have the relationship we are after:

$$a^2 + b^2 = c^2$$

PRACTICE PROBLEMS

1. Solve for x:



EXAMPLE 1 Solve for x in the right triangle in Figure 3.

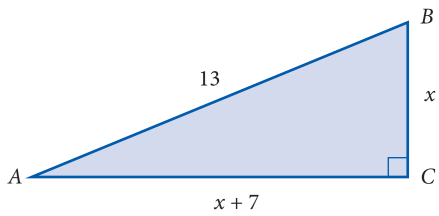


FIGURE 3

SOLUTION Applying the Pythagorean Theorem gives us a quadratic equation to solve.

$$
\begin{aligned}
(x + 7)^2 + x^2 &= 13^2 && \\
x^2 + 14x + 49 + x^2 &= 169 && \text{Expand } (x+7)^2 \text{ and } 13^2 \\
2x^2 + 14x + 49 &= 169 && \text{Combine similar terms} \\
2x^2 + 14x - 120 &= 0 && \text{Add } -169 \text{ to both sides} \\
x^2 + 7x - 60 &= 0 && \text{Divide both sides by 2} \\
(x - 5)(x + 12) &= 0 && \text{Factor the left side} \\
x - 5 = 0 \quad \text{or} \quad x + 12 &= 0 && \text{Set each factor to 0} \\
x = 5 \quad \text{or} \quad x &= -12 &&
\end{aligned}
$$

Our only solution is $x = 5$. We cannot use $x = -12$ since x is the length of a side of triangle ABC and therefore cannot be negative.

NOTE The lengths of the sides of the triangle in Example 1 are 5, 12, and 13. Whenever the three sides in a right triangle are natural numbers, those three numbers are called a *Pythagorean triple*.

2. Another chair lift has a vertical rise of 960 feet and the length of the chair is 4,520 feet. To the nearest foot, find the horizontal distance covered by a person riding the lift.

EXAMPLE 2 The vertical rise of the Forest Double chair lift (Figure 4) is 1,170 feet and the length of the chair lift as 5,750 feet. To the nearest foot, find the horizontal distance covered by a person riding this lift.

FIGURE 4

Answer
1. 8

SOLUTION Figure 5 is a model of the Forest Double chair lift. A rider gets on the lift at point A and exits at point B. The length of the lift is AB.

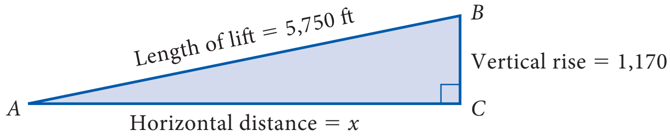


FIGURE 5

To find the horizontal distance covered by a person riding the chair lift we use the Pythagorean Theorem:

$$5{,}750^2 = x^2 + 1{,}170^2 \quad \text{Pythagorean Theorem}$$

$$33{,}062{,}500 = x^2 + 1{,}368{,}900 \quad \text{Simplify squares}$$

$$x^2 = 33{,}062{,}500 - 1{,}368{,}900 \quad \text{Solve for } x^2$$

$$x^2 = 31{,}693{,}600 \quad \text{Simplify the right side}$$

$$x = \sqrt{31{,}693{,}600}$$

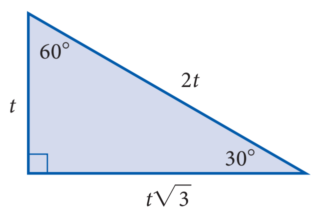
$$x = 5{,}630 \text{ ft} \quad \text{To the nearest foot}$$

A rider getting on the lift at point A and riding to point B will cover a horizontal distance of approximately 5,630 feet.

Before leaving the Pythagorean Theorem we should mention something about Pythagoras and his followers, the Pythagoreans. They established themselves as a secret society around the year 540 B.C. The Pythagoreans kept no written record of their work; everything was handed down by spoken word. Their influence was not only in mathematics, but also in religion, science, medicine, and music. Among other things, they discovered the correlation between musical notes and the reciprocals of counting numbers, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and so on. In their daily lives they followed strict dietary and moral rules to achieve a higher rank in future lives. The British philosopher Bertrand Russell has referred to Pythagoras as "intellectually one of the most important men that ever lived."

B The 30°–60°–90° Triangle

In any right triangle in which the two acute angles are 30° and 60°, the longest side (the hypotenuse) is always twice the shortest side (the side opposite the 30° angle), and the side of medium length (the side opposite the 60° angle) is always $\sqrt{3}$ times the shortest side (Figure 6).



30° − 60° − 90°

FIGURE 6

Answer
2. 4,417 feet

NOTE The shortest side t is opposite the smallest angle 30°. The longest side $2t$ is opposite the largest angle 90°. To verify the relationship between the sides in this triangle, we draw an equilateral triangle (one in which all three sides are equal) and label half the base with t (Figure 7).

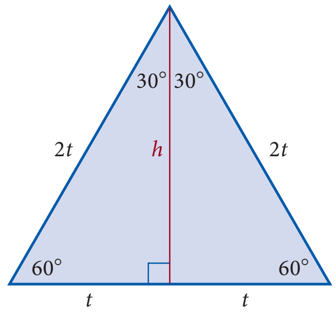


FIGURE 7

The altitude h bisects the base. We have two 30°–60°–90° triangles. The longest side in each is $2t$. We find that h is $t\sqrt{3}$ by applying the Pythagorean Theorem.

$$\begin{aligned} t^2 + h^2 &= (2t)^2 \\ h &= \sqrt{4t^2 - t^2} \\ &= \sqrt{3t^2} \\ &= t\sqrt{3} \end{aligned}$$

3. If the hypotenuse of a 30° – 60° – 90° triangle is 12, find the other 2 sides.

EXAMPLE 3 If the shortest side of a 30°–60°–90° triangle is 5, find the other two sides.

SOLUTION The longest side is 10 (twice the shortest side), and the side opposite the 60° angle is $5\sqrt{3}$ (Figure 8).

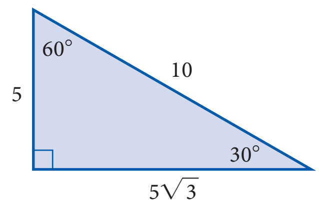


FIGURE 8

Algebra Review: Rationalizing the Denominator

Radical expressions that are in simplified form are generally easier to work with. A radical expression is in simplified form if it has three special characteristics.

Definition

A radical expression is in **simplified form** if

1. There are no perfect squares that are factors of the quantity under the square root sign, no perfect cubes that are factors of the quantity under the cube root sign, and so on. We want as little as possible under the radical sign.
2. There are no fractions under the radical sign.
3. There are no radicals in the denominator.

Answer

3. $6, 6\sqrt{3}$

A radical expression that has these three characteristics is said to be in simplified form. As we will see, simplified form is not always the least complicated expression. In many cases, the simplified expression looks more complicated than the original expression. The important thing about simplified form for radicals is that simplified expressions are easier to work with.

The tools we will use to put radical expressions into simplified form are the properties of radicals. We list the properties again for clarity.

Properties of Radicals

If a and b represent any two nonnegative real numbers, then it is always true that

1. $\sqrt{a}\,\sqrt{b} = \sqrt{a \cdot b}$
2. $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ $\qquad b \neq 0$
3. $\sqrt{a}\,\sqrt{a} = (\sqrt{a})^2 = a$ $\qquad$ This property comes directly from the definition of radicals

The following examples illustrate how we put a radical expression into simplified form using the three properties of radicals. Although the properties are stated for square roots only, they hold for all roots. [Property 3 written for cube roots would be $\sqrt[3]{a}\,\sqrt[3]{a}\,\sqrt[3]{a} = (\sqrt[3]{a})^3 = a$.]

EXAMPLE 4 Put $\sqrt{\dfrac{1}{2}}$ into simplified form.

SOLUTION The expression $\sqrt{\frac{1}{2}}$ is not in simplified form because there is a fraction under the radical sign. We can change this by applying Property 2 for radicals:

$$\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} \qquad \text{Property 2 for radicals}$$

$$= \frac{1}{\sqrt{2}} \qquad \sqrt{1} = 1$$

The expression $\frac{1}{\sqrt{2}}$ is not in simplified form because there is a radical sign in the denominator. If we multiply the numerator and denominator of $\frac{1}{\sqrt{2}}$ by $\sqrt{2}$, the denominator becomes $\sqrt{2} \cdot \sqrt{2} = 2$:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{\mathbf{2}}}{\sqrt{\mathbf{2}}} \qquad \text{Multiply numerator and denominator by } \sqrt{2}$$

$$= \frac{\sqrt{2}}{2} \qquad \begin{matrix} 1 \cdot \sqrt{2} = \sqrt{2} \\ \sqrt{2} \cdot \sqrt{2} = \sqrt{4} = 2 \end{matrix}$$

If we check the expression $\frac{\sqrt{2}}{2}$ against our definition of simplified form for radicals, we find that all three rules hold. There are no perfect squares that are factors of 2. There are no fractions under the radical sign. No radicals appear in the denominator. The expression $\frac{\sqrt{2}}{2}$, therefore, must be in simplified form.

Note Simplified form for radicals is the form that we work toward when simplifying radicals. The properties of radicals are the tools we use to get us to simplified form.

4. Put $\sqrt{\dfrac{1}{3}}$ in simplified form.

Answer

4. $\frac{\sqrt{3}}{3}$

5. Write $\sqrt{\frac{3}{5}}$ in simplified form.

6. Another ladder is leaning against a wall. The top of the ladder is 7 feet above the ground, and the bottom of the ladder makes an angle of 60° with the ground. How long is the ladder and how far from the wall is the bottom of the ladder?

Answers

5. $\frac{\sqrt{15}}{5}$ 6. 4.04 ft., 8.08 ft.

EXAMPLE 5

Write $\sqrt{\frac{2}{3}}$ in simplified form.

SOLUTION We proceed as we did in Example 4:

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} \quad \text{Use Property 2 to separate radicals}$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{\mathbf{3}}}{\sqrt{\mathbf{3}}} \quad \text{Multiply by } \frac{\sqrt{3}}{\sqrt{3}} \text{ to remove the radical from the denominator}$$

$$= \frac{\sqrt{6}}{3} \quad \begin{array}{l} \sqrt{2} \cdot \sqrt{3} = \sqrt{6} \\ \sqrt{3} \cdot \sqrt{3} = \sqrt{9} = 3 \end{array}$$

EXAMPLE 6

A ladder is leaning against a wall. The top of the ladder is 4 feet above the ground and the bottom of the ladder makes an angle of 60° with the ground (Figure 9). How long is the ladder, and how far from the wall is the bottom of the ladder?

SOLUTION The triangle formed by the ladder, the wall, and the ground is a 30°–60°–90° triangle. If we let x represent the distance from the bottom of the ladder to the wall, then the length of the ladder can be represented by $2x$. The distance from the top of the ladder to the ground is $x\sqrt{3}$, since it is opposite the 60° angle (Figure 10). It is also given as 4 feet. Therefore,

$$x\sqrt{3} = 4$$

$$x = \frac{4}{\sqrt{3}}$$

$$= \frac{4\sqrt{3}}{3} \quad \text{Rationalize the denominator by multiplying the numerator and denominator by } \sqrt{3}.$$

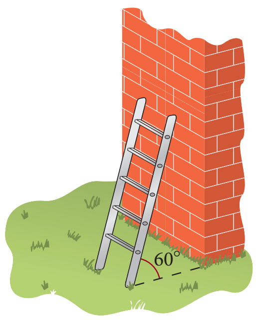


FIGURE 9

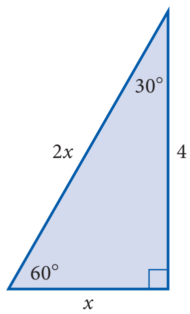


FIGURE 10

The distance from the bottom of the ladder to the wall, x, is $\frac{4\sqrt{3}}{3}$ feet, so the length of the ladder, $2x$, must be $\frac{8\sqrt{3}}{3}$ feet. Note that these lengths are given in exact values. If we want a decimal approximation for them, we can replace $\sqrt{3}$ with 1.732 to obtain

$$\frac{4\sqrt{3}}{3} \approx \frac{4(1.732)}{3} = 2.309 \text{ ft}$$

$$\frac{8\sqrt{3}}{3} \approx \frac{8(1.732)}{3} = 4.619 \text{ ft}$$

CALCULATOR NOTE On a scientific calculator, this last calculation could be done as follows:

$$8 \ \boxed{\times} \ 3 \ \boxed{\sqrt{\ }} \ \boxed{\div} \ 3 \ \boxed{=}$$

On a graphing calculator, the calculation is done like this:

$$8 \ \boxed{\times} \ \boxed{\sqrt{\ }} \ \boxed{(} \ 3 \ \boxed{)} \ \boxed{\div} \ 3 \ \boxed{\text{ENTER}}$$

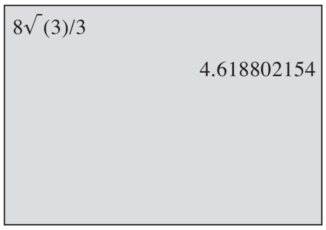


FIGURE 11

Some graphing calculators use parentheses with certain functions, such as the square root function. For example, the TI-83 will automatically insert a left parenthesis, so TI-83 users should skip this key. Other models do not require them. For the sake of clarity, we will often include parentheses throughout this book. You may be able to omit one or both parentheses with your model.

C The 45°–45°–90° Triangle

If the two acute angles in a right triangle are both 45°, then the two shorter sides (the legs) are equal and the longest side (the hypotenuse) is $\sqrt{2}$ times as long as the shorter sides. That is, if the shorter sides are of length t, then the longest side has length $t\sqrt{2}$ (Figure 12).

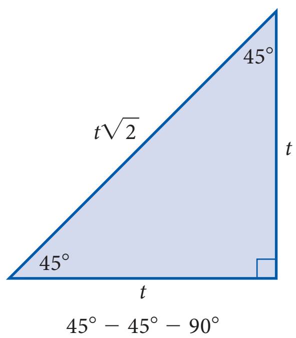


FIGURE 12

To verify this relationship, we simply note that if the two acute angles are equal, then the sides opposite them are also equal. We apply the Pythagorean Theorem to find the length of the hypotenuse.

$$\begin{aligned} \text{hypotenuse} &= \sqrt{t^2 + t^2} \\ &= \sqrt{2t^2} \\ &= t\sqrt{2} \end{aligned}$$

7. If the rope in Example 7 was 12 feet long, find the length of the tent pole.

EXAMPLE 7 A 10-foot rope connects the top of a tent pole to the ground. If the rope makes an angle of 45° with the ground, find the length of the tent pole (Figure 13).

SOLUTION Assuming that the tent pole forms an angle of 90° with the ground, the triangle formed by the rope, tent pole, and the ground is a 45° – 45° – 90° triangle (Figure 14).

If we let x represent the length of the tent pole, then the length of the rope, in terms of x, is $x\sqrt{2}$. It is also given as 10 feet. Therefore,

$$x\sqrt{2} = 10$$

$$x = \frac{10}{\sqrt{2}} = 5\sqrt{2}$$

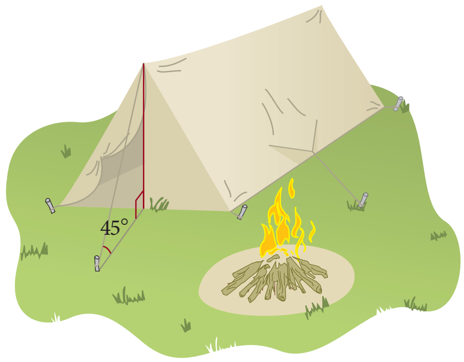


FIGURE 13

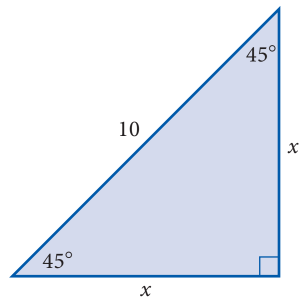


FIGURE 14

The length of the tent pole is $5\sqrt{2}$ feet. Again, $5\sqrt{2}$ is the exact value of the length of the tent pole. To find a decimal approximation, we replace $\sqrt{2}$ with 1.414 to obtain

$$5\sqrt{2} \approx 5(1.414) = 7.07 \text{ ft}$$

Getting Ready for Class

After reading through the preceding section, respond in your own words and in complete sentences.

1. Describe how you would put $\sqrt{\frac{1}{2}}$ in simplified form.
2. What is simplified form for an expression that contains a square root?
3. What does it mean to rationalize the denominator in an expression?
4. Why is it important to recognize 30° – 60° – 90° and 45° – 45° – 90° triangles?

Answer
7. $6\sqrt{2} \approx 8.49$ ft.